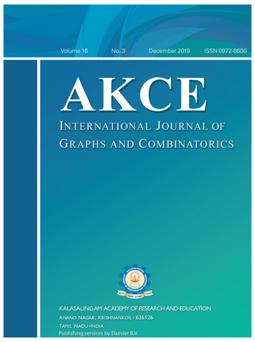

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More connections between the matching polynomial and the chromatic polynomial

Beatriz Carely Luna-Olivera[a,1], Criel Merino[b,2], Marcelino Ramírez-Ibáñez[a,*,1]

[a] Instituto de Agroingeniería, Universidad del Papaloapan, Av. Ferrocarril S/N Loma Bonita, 68400, Oaxaca, Mexico
[b] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Área de la Investigación científica, Circuito Exterior, C.U., Coyoacán, 04510, México D.F., Mexico

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Abstract

The connection between the matching polynomial and the chromatic polynomial for triangle-free graphs was revealed in the work of Farrell and Whitehead. We extend this result to all graph by mirroring the corresponding result of Godsil and Gutman for the acyclic polynomial and the characteristic polynomial. We also reintroduce the clique polynomial of Farrell as an evaluation of the U-polynomial of Noble and Welsh, which also happens to contain the matching and chromatic polynomials.

Keywords: Matching polynomial; Chromatic polynomial; F-polynomial; U-polynomial

1. Introduction

Farrell introduced a class of graph polynomials in [1] in the following way. Let $\mathcal{F}$ be a family of connected graphs. To each member α in the family we associated a weight w_α, usually a variable. For a graph G, an $\mathcal{F}$-*subgraph* is a subgraph of G with all its connected components belonging to $\mathcal{F}$. The $\mathcal{F}$-subgraph is said to be *proper* if all the components have size at least three. The $\mathcal{F}$-subgraph is a *cover* if it is a spanning subgraph of G. To the $\mathcal{F}$-subgraph F with connected components $\alpha_1, \dots, \alpha_k$ we associate the monomial $\Pi(F) = \prod_{i=1}^{k} w_{\alpha_i}$, with not necessarily all the α_i different. Now, we can define the $\mathcal{F}$-polynomial of G to be

$$\mathcal{F}(G; \mathbf{w}) = \sum_{F} \Pi(F), \tag{1}$$

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* Corresponding author.
E-mail addresses: bcluna@unpa.edu.mx (B.C. Luna-Olivera), merino@matem.unam.mx (C. Merino), mramirez@unpa.edu.mx (M. Ramírez-Ibáñez).
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where the sum is over all $\mathcal{F}$-subgraphs that are covers of G. Some of the most interesting examples are for the families $\{P_n\}$, $\{S_n\}$, $\{T_n\}$, $\{C_n\}$, and $\{K_n\}$ for $n \in \mathbb{Z}_{>0}$, that is, the families of paths, starts, trees, cycles and complete graphs with n vertices. We assume that C_1 is the empty graph with one vertex and C_2 is the complete graph with two vertices.

We concentrate here in two of these polynomials, the circuit polynomial, $C(G; \mathbf{w})$, where we take the family $\{C_n\}$ and the weight of a cycle C_n is w_n; and the clique polynomial, $K(G; \mathbf{w})$, where we take the family $\{K_n\}$ and the weight of a complete graph K_n is w_n. By allowing the $\mathcal{F}$ family to be the empty graphs $\{\bar{K_n}\}$ with n vertices, where each $\bar{K_n}$ has weight w_n, we obtain a polynomial which gives the same information as $K(G; \mathbf{w})$, because $K(\bar{G}; \mathbf{w}) = \bar{K}(G; \mathbf{w})$, but that is more easily related to the chromatic polynomial.

Now, we recover some classic graph polynomials. *The characteristic polynomial* of a graph G, $\phi(G; x)$ is defined as the characteristic polynomial of the adjacency matrix of G. At first sight, this is an algebraic rather than a combinatorial definition. But a deeper inspection gives the following result of Farrell [1].

$$\phi(G, x) = C(G; w_1 = x, w_2 = -1, w_k = -2 \text{ if } k \geq 3), \tag{2}$$

Now, another polynomial enters the picture. *The matching defect polynomial*, or *acyclic polynomial*, $m(G, x)$ of a graph G with n vertices is defined as

$$m(G, x) = \sum_{i=0}^{\lceil n/2 \rceil} (-1)^i m_i x^{n-2i}, \tag{3}$$

where m_i is the number of matchings with i edges. Now, the following result appears in [2]

$$\phi(G, x) = m(G, x) + \sum_{\text{proper } F} (-2)^{|F|} m(G \setminus F, x), \tag{4}$$

where the sum is over all proper $\{C_n\}$-subgraphs of G and for a subgraph F, $G \setminus F$ denotes the subgraph of G obtained by deleting from G the vertices of F. By Möbius inversion we get

$$m(G, x) = \phi(G, x) + \sum_{\text{proper } F} 2^{|F|} \phi(G \setminus F, x). \tag{5}$$

Also notice that $m(G, x) = C(G; w_1 = x, w_2 = 1, w_k = 0 \text{ if } k \geq 3)$.

2. The chromatic and matching polynomial

Let us mimic the previous line of thought with another pair of graph polynomials. We follow the exposition of Farrell in [3]. This time, we take as the family $\mathcal{F}$, the complete graphs $\{K_n\}$. The corresponding polynomial is the clique polynomial that was studied in [4]. Now, we introduce the other pair of graph polynomials.

The (2-variable) *matching polynomial* $M(G; x, y)$ of a graph with n vertices is very similar to the matching defect polynomial and is defined as

$$M(G; x, y) = \sum_{i=0}^{\lceil n/2 \rceil} m_i x^{n-2i} y^i, \tag{6}$$

thus, $m(G; x) = M(G; x, -1)$. Also, $M(G; 1, y)$ is the generating function of matchings by size. *The chromatic polynomial*, $\chi(G; x)$ is defined as the function whose value at integer $i \geq 0$ is the number of proper colourings with i colours of the graph G. It is a classical result that this is a polynomial, see [5]. The chromatic polynomial has an expression as a generating function of stable sets, or independent sets, as follows.

$$\chi(G, x) = \sum_F x \cdot (x-1) \cdots (x - |F| + 1), \tag{7}$$

where the sum is over all $\{\bar{K_n}\}$-subgraph covers, that is over the partition of the vertices of G as stable sets. See [6] for more details. The polynomials $x(x-1)(\cdots)(x-k+1)$ that we denote $x_{(k)}$, for $k \geq 0$, form a basis for the ring of polynomials $\mathbb{R}[x]$.

In a similar fashion as in the previous section

$$\chi(G; x) = K(\bar{G}; w_k = w)|_{w^k \to x_{(k)}} \tag{8}$$

$$M(G, x, y) = K(G; w_1 = x, w_2 = y, w_k = 0 \text{ if } k \geq 3). \tag{9}$$

The substitution $w^k \to x_{(k)}$ consists in changing the elements of the basis w^k of $\mathbb{R}[w]$ by the elements of the basis $x_{(k)}$ of $\mathbb{R}[x]$.

A classical theorem relates the chromatic and the matching polynomial, see [3].

Theorem 1. *If G is triangle-free, then*

$$\chi(\bar{G}, x) = M(G; w, w)|_{w^k \to x_{(k)}}. \tag{10}$$

Before we generalize this by mimicking the result in the introduction, we need to make some observations: note that any cover F can be split into two disjoint sets $F = F_1 \cup F_2$, where F_1, F_2 contain only improper and proper elements of F, respectively. If $|F_1|$ and $|F_2|$ are the number of improper and proper components of F, respectively, and $w_k = w$ for $k \geq 1$, then

$$\Pi(F) = w^{|F_1|} \cdot w^{|F_2|}.$$

Another observation is that

$$\sum_F \Pi(F) = \sum_{\text{improper } F} w^{|F|} + \sum_{F=F_1\cup F_2, F_2 \neq \emptyset} w^{|F_1|} \cdot w^{|F_2|}, \tag{11}$$

where the covers in the first summand only contain improper elements of $\mathcal{F}$.

Theorem 2. *For a general graph G, we have that*

$$\chi(\bar{G}, x) = \left(M(G, w, w) + \sum_{\text{proper } F} w^{|F|} M(G \setminus F, w, w) \right) |_{w^k \to x_{(k)}}, \tag{12}$$

where the sum is over all proper $\{K_n\}$-subgraphs F of G and $G \setminus F$ denotes the subgraph of G obtained by deleting from G the vertices of F.

Proof. Following [3], $\chi(\bar{G}; x) = \sum_{k=0}^n c_k x_{(n-k)}$ where c_k is the number of partitions of the vertices of $\bar{G}$ in $n-k$ chromatic classes. Each $n-k$ chromatic class is a cover of G where every connected component of the cover is a complete graph. This gives a bijection between chromatic classes of $\bar{G}$ and F-covers of G and we have that c_k is the number of F-covers of G with $n-k$ components, so

$$\begin{aligned}
\chi(\bar{G}, x) &= \sum_{k=0}^n c_k w^{n-k} |_{w^k \to x_{(k)}} = \sum_{F\text{-coverof } G} w^{|F|} |_{w^k \to x_{(k)}} \\
&= \left(M(G, w, w) + \sum_{F=F_1\cup F_2} w^{|F_1|} w^{|F_2|} \right) |_{w^k \to x_{(k)}} \\
&= \left(M(G, w, w) + \sum_{\text{proper } F} w^{|F|} M(G \setminus F, w, w) \right) |_{w^k \to x_{(k)}}
\end{aligned}$$

where the third equality is obtained by applying Eq. (11). □

Again, by Möbius inversion we get the following corollary.

Corollary 1. *For a general graph G, we have that*

$$M(\bar{G}, w, w) = \chi(G, x)|_{x_{(k)} \to w^k} + \sum_{\text{proper } F} (-1)^{|F|} w^{|F|} \left(\chi(G \setminus F, x)|_{x_{(k)} \to w^k} \right), \tag{13}$$

where the sum is over all proper $\{\bar{K}_n\}$-subgraphs F of G.

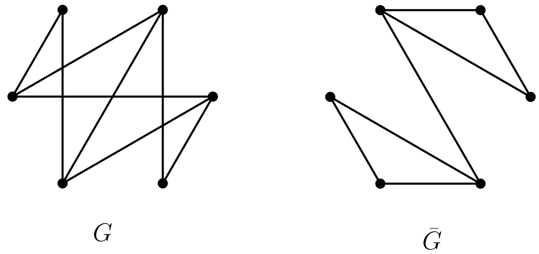

Fig. 1. A graph G and its complement $\bar{G}$.

2.1. Example

Let G be the graph shown in Fig. 1, note that its complement $\bar{G}$ has two complete graphs, which are disjoint triangles that we shall denote T_1 and T_2, $\bar{G} \setminus T_1 = T_2$ and $\bar{G} \setminus T_2 = T_1$, and $G \setminus \{T_1, T_2\}$ is empty. The chromatic polynomial of G is $\chi(G, x) = x_{(6)} + 7x_{(5)} + 13x_{(4)} + 7x_{(3)} + x_{(2)}$, and applying Theorem 2 we find that $\chi(G, x) = (M(\bar{G}, w, w) + wM(T_1, w, w) + wM(T_2, w, w) + w^2M(\emptyset, w, w))|_{w^k \to x_{(k)}} = (w^6 + 7w^5 + 11w^4 + w^3 + 2w(w^3 + 3w^2) + w^2(1))|_{w^k \to x_{(k)}} = (w^6 + 7w^5 + 13w^4 + 7w^3 + w^2)|_{w^k \to x_{(k)}}$.

3. The U-polynomial

The U-polynomial was defined in [7] and we follow the presentation there. Given a spanning subgraph (V, A) of a graph $G = (V, E)$, we denote by $\kappa(A)$ its number of components and by $r(A)$ the rank of the subset of edges A, that is $|V| - \kappa(A)$. Then *the U-polynomial* is defined as

$$U(G; \mathbf{w}, y) = \sum_{A \subseteq E} w_{n_1} \cdots w_{n_{\kappa(A)}} (y-1)^{|A|-r(A)}, \tag{14}$$

where n_i is the number of vertices in the i-connected component of (V, A). It is known that the U-polynomial contains as specialization the Tutte polynomial, the stability polynomial, the chromatic polynomial and the matching polynomial. Also, if the graph G is a simple graph with n vertices, the number of clique subgraphs of G can be read off the U-polynomial as this number is the coefficient of the term $w_k w_1^{n-k} y^{\binom{k}{2}-k+1}$. The proof in [7] uses two results. The first one is that a simple graph G is a clique if and only if the coefficient of $y^{\binom{k}{2}-k+1}$ in $T(G; 1, y)$ is positive (actually equal to 1). The polynomial $T(G; 1, y)$ is an evaluation of the Tutte polynomial and has many different interpretations, see [8,9]. The other is a different expression for the U-polynomial as

$$U(G; \mathbf{w}, y) = \sum_{\Pi} w(\Pi) T(G_1; 1, y) \cdots T(G_k; 1, y), \tag{15}$$

where the sum of over all partitions of the vertices of G, $\Pi = V_1 \cup \cdots \cup V_k$, such that the induce subgraph of $G_i = G[V_i]$ is a connected graph for all i and the monomial $w(\Pi)$ is defined by $w(\Pi) = w_{|V_1|} \cdots w_{|V_k|}$.

It is not difficult to get the polynomial of cliques from this expression.

Theorem 3. *The clique polynomial of a simple graph can be obtained from the U-polynomial by a sequence of substitutions.*

Proof. For a simple graph G, consider the term $w(\Pi)T(G_1; 1, y) \cdots T(G_k; 1, y)$ for a connected partition Π in the expression (15) for the U-polynomial. After the substitution $w_k \to x_k z^{\binom{k}{2}-k+1}$ for $k \geq 2$, $w_1 = x_1$ and $y \to 1/z$, the new polynomial in z will have an independent term $x(\Pi)$ if and only if each component in the connected partition Π is a complete graph. Now, by evaluating at $z = 0$ we obtain the clique polynomial. □

The strong U-polynomial was defined in [10] and it is denoted by $\overline{U}$, which is a polynomial in countably many commuting variables $z_{i,j}$ where $i \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{\geq 0}$.

$$\overline{U}_G(\mathbf{z}) = \sum_{A \subseteq E} z_{n_1,m_1-n_1+1} z_{n_2,m_2-n_2+1} \cdots z_{n_{k(G|A)},m_{k(G|A)}-n_{k(G|A)}+1}, \tag{16}$$

where n_i and m_i are, respectively, the number of vertices and edges in the ith connected component of $G|A$.

An equivalent polynomial is defined by Farrell in [1] by using $\mathcal{F}$-polynomial. The polynomial is called *subgraph polynomial* and is denoted $S(G; \mathbf{w})$. To define it we take $\mathcal{F}$ to be the family $\{G_{n,m}\}$ of all connected graphs with n vertices and m edges, for $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{\geq 0}$. Here the weight of a graph $G_{n,m}$ is $w_{n,m}$. Clearly, the polynomials are equivalent as we obtain one from the other by making the substitution $w_{n,m} = z_{n,m-n+1}$.

Thus, the subgraph polynomial contains the clique polynomial as an specialization. Actually, the subgraph polynomial is quite general because it is proved in [10] that the strong U-polynomial is equivalent to the natural extensions of Stanley's Tutte symmetric function, see [11], and the polychromate, introduced originally by Brylawski in [12]. Then, the subgraph polynomial contains these as specializations in the sense described in [10], in particular, the Tutte and chromatic polynomial.

4. Conclusion

Using the Farrell polynomial with the clique family we have shown a connection between matching polynomial and chromatic polynomial for simple finite graphs without restrictions, we have also shown a relationship between the U-polynomial and the clique polynomial. We have found the clique polynomial from two different approaches. All these polynomials are specializations of the subgraph polynomial of Farrell.

This approach has been considered in other works, most recently by Zhang and Dong in [13] where they relate the independence polynomial and a chromatic polynomial for hypergraphs.

It is not clear to us if there exist another connection between the U-polynomial and the F-polynomial for other interesting families of graphs, for example the circuit polynomial.

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